Mathematical Finance
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Assignment 4

We fix throughout a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which we are given a filtration $\mathbb{F}$, unless otherwise stated.

Class $L\log(L)$

Let $(X_n)_{n\in\mathbb{N}}$ be a discrete-time $(\mathbb{F},\mathbb{P})$–sub-martingale (so obviously here $\mathbb{F}$ is a discrete filtration) and define

$$\overline{X}_n := \max_{i\in\{0,\dots,n\}} X_i,\ n\in\mathbb{N}.$$

Define then for any $(x,y)\in\mathbb{R}\times\mathbb{R}_+^\star$

$$x^\star := \max\{x,0\},\ \log^+(y) := \max\{\log(y),0\}.$$

1) let Y be a non-negative random variable, and let $f:\mathbb{R}_+\longrightarrow\mathbb{R}$ a non-decreasing map, which is piecewise C^1 on $\mathbb{R}_+$. Show that for any $c\geq 0$
$$\mathbb{E}^{\mathbb{P}}[f(Y)] \leq f(c) + \int_c^{+\infty} f'(\lambda)\mathbb{P}[Y\geq\lambda]\mathrm{d}\lambda.$$

2) Show that for any positive constants a and b
$$a\log(b) \leq a\log^+(a) + b\mathrm{e}^{-1}.$$

3) Show that for any $M>1$
$$\mathbb{E}^{\mathbb{P}}\big[\min\big\{\overline{X}_n^+, M\big\}\big] \leq 1 + \mathbb{E}^{\mathbb{P}}\Big[X_n^+\log\big(\min\big\{\overline{X}_n^+, M\big\}\big)\Big],$$
and deduce
$$\mathbb{E}^{\mathbb{P}}\big[\overline{X}_n^+\big] \leq \frac{1+\mathbb{E}^{\mathbb{P}}\big[X_n^+\log^+(X_n^+)\big]}{1-\mathrm{e}^{-1}}.$$
Extend this result to right-continuous $(\mathbb{F},\mathbb{P})$–sub-martingales in continuous-time.

4) Show that if $\sup_{n\in\mathbb{N}}|X_n|\leq Y$ for some non-negative random variable Y such that $\mathbb{E}^{\mathbb{P}}[Y]<+\infty$, then $(X_n)_{n\in\mathbb{N}}$ is $\mathbb{P}$–uniformly integrable.

5) Show that if $(X_n)_{n\in\mathbb{N}}$ is an $(\mathbb{F},\mathbb{P})$-martingale such that
$$\sup_{n\in\mathbb{N}}\mathbb{E}^{\mathbb{P}}\big[|X_n|\log^+|X_n|\big] < +\infty,$$
then $(X_n)_{n\in\mathbb{N}}$ converges in $\mathbb{L}^1(\mathbb{R},\mathcal{F},\mathbb{P})$.

A martingale with non-integrable running maximum

The previous exercise shows that for any right-continuous $(\mathbb{F},\mathbb{P})$-martingale X and any $0\leq t$

$$\mathbb{E}^{\mathbb{P}}\bigg[\sup_{s\in[0,t]}|X_s|\bigg] \leq (1-\mathrm{e}^{-1})^{-1}\Big(1+\mathbb{E}^{\mathbb{P}}\big[|X_t|\log^+\big(|X_t|\big)\big]\Big).$$

The goal of this exercise is to prove that one cannot improve the right-hand side in the sense that $\mathbb{E}^{\mathbb{P}}\big[\sup_{s\in[0,t]}|X_s|\big]$ can be infinite in general.

1) Fix some map $h : (0,1) \longrightarrow \mathbb{R}$ which is continuous, non-negative, decreasing, and Lebesgue-integrable on $(0,1)$ and define its running average
$$\bar{h}(t) := \frac{1}{t}\int_0^t h(s)\mathrm{d}s,\ t > 0.$$
Let U be a random variable whose $\mathbb{P}$-distribution is uniform on $(0,1)$. Define the process
$$X_t := \bar{h}(1-t)\mathbf{1}_{\{t<1-U\}} + h(U)\mathbf{1}_{\{t\geq 1-U\}},\ t \geq 0.$$
Show that X is a martingale under $\mathbb{P}$ for its natural filtration $\mathbb{F}$, and that
$$X_1 = h(U),\ X_1^\star := \sup_{s\in[0,1]} |X_s| \geq \bar{h}(U).$$
Hint: it would be useful to prove first here that $\mathcal{F}_t = \sigma\big(U\mathbf{1}_{\{1-U\leq t\}}\big)$, $t \geq 0$.

2) Find an example of h which is Lebesgue-integrable on $(0,1)$ but such that $\mathbb{E}^{\mathbb{P}}[X_1^\star] = +\infty$.

Stochastic integration versus Lebesgue–Stieltjes integration

Consider the setting of Example 6.3.2 in the lecture notes. Fix a sequence $(t_n)_{n\in\mathbb{N}}$ of distinct times in $[0,1]$, and let us be given a sequence $(U_n)_{n\in\mathbb{N}}$ of $\mathbb{P}$-independent random variables such that $\mathbb{P}[U_n = 1] = \mathbb{P}[U_n = -1] = 1/2$, $n \in \mathbb{N}$. We then define
$$V_t := \sum_{n=0}^{+\infty} \mathbf{1}_{\{t_n\leq t\}} \frac{U_n}{(n+1)^2},\ t \geq 0.$$

1) Show that V is well-defined, that it is a càdlàg pure jump process, which is constant for $t \geq 1$, and whose $\mathbb{P}$-variation is also finite.

   Take $\mathbb{F}$ as being the $\mathbb{P}$-completed natural filtration of V.

2) Show that V is an $(\mathbb{F},\mathbb{P})$-martingale which is in addition $\mathbb{P}$–square-integrable.

3) Define
$$\xi_t := \sum_{n=0}^{+\infty} \mathbf{1}_{\{t_n=t\}}(n+1),\ t \geq 0.$$
Show that ξ is $\mathbb{F}$-predictable and that $\xi \in \mathcal{L}^1(V,\mathbb{F},\mathbb{P})$ (you may want to use here Lemma 6.7.12).

4) Define
$$Y_t := \int_0^t \xi_s \mathrm{d}V_s,\ t \geq 0.$$
Show that
$$Y_t = \sum_{n=0}^{\infty} \mathbf{1}_{\{t_n\leq t\}} \frac{U_n}{n+1},\ t \geq 0,$$
and justify in particular why the sum on the right-hand side converges $\mathbb{P}$–almost surely. Deduce that Y does not have $\mathbb{P}$–finite variation and that
$$\int_0^1 |\xi_s||\mathrm{d}V|_s + \infty.$$
Is ξ integrable with respect to V in the Lebesgue–Stieltjes sense?

5) Take the specific example of $t_n := \frac{n}{n+1}$, $n \in \mathbb{N}$. Prove then that ξ is bounded on $[0, t]$ for any time $t < 1$ so that $\int_0^t \xi_s \mathrm{d}V_s$ is well-defined on the individual sample paths, and one can define the 'improper' Lebesgue–Stieltjes integral

$$\int_0^1 \xi_s \mathrm{d}V_s := \lim_{t \uparrow 1} \int_0^t \xi_s \mathrm{d}V_s,$$

where convergence is in the $\mathbb{P}$–almost sure sense.

6) Can you find another choice for $(t_n)_{n\in\mathbb{N}}$ such that $\int_s^t |\xi_u||\mathrm{d}V|_u$ is infinite on all non-empty intervals (s, t)?

Hint: you may want to have a look at Weyl's equidistribution theorem.